7.4: Derivatives, Integrals, and Products of Transforms

Example 1. Find $X(s)$ in differential equation

$$x'' + x = \cos t.$$

Definition 1. (The Convolution of Two Functions)
The **convolution** $f * g$ of the piecewise continuous functions f and g is defined for $t \geq 0$ as follows:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Example 2. Find the convolution of $\cos t$ and $\sin t$.

Theorem 1. (The Convolution Property)
Suppose that $f(t)$ and $g(t)$ are piecewise continuous for $t \geq 0$ and that $|f(t)|$ and $|g(t)|$ are bounded by Me^{ct} as $t \to \infty$. Then the Laplace transform of the convolution $f(t) * g(t)$ exists for $s > c$; moreover,

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

and

$$\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f(t) * g(t).$$

Example 3. Find $\mathcal{L}^{-1}\{\frac{2}{(s-1)(s^2+4)}\}$.

Theorem 2. (Differentiation of Transforms)
If $f(t)$ is piecewise continuous for $t \geq 0$ and $|f(t)| \leq Me^{ct}$ as $t \to \infty$, then

$$\mathcal{L}\{-tf(t)\} = F'(s)$$

for $s > c$. Equivalently,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}.$$

Repeated applications gives, for $n = 1, 2, 3, \ldots$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s).$$

Example 4. Find $\mathcal{L}\{t^2 \sin kt\}$.

Example 5. Find $\mathcal{L}^{-1}\{\tan^{-1}(1/s)\}$.

Example 6. Find the solution to the initial value problem

$$tx'' + x' + tx = 0; \quad x(0) = 1, \quad x'(0) = 0.$$

Theorem 3. (Integration of Transforms)
Suppose that $f(t)$ is piecewise continuous for $t \geq 0$, that $\lim_{t \to 0^+} \frac{f(t)}{t}$ exists and is finite, and that $|f(t)| \leq Me^{ct}$ as $r \to \infty$. Then

$$\mathcal{L}\{\frac{f(t)}{t}\} = \int_s^\infty F(\sigma) d\sigma$$

for $s > c$. Equivalently,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t\mathcal{L}^{-1}\{\int_s^\infty F(\sigma) d\sigma\}.$$

Example 7. Find $\mathcal{L}\{\frac{\sinh t}{t}\}$.

Example 8. Find $\mathcal{L}^{-1}\{\frac{2s}{(s^2-1)^2}\}$

Homework. 1-33 (odd)